- **Factorization and Evolution**
- **Resummation: an Example**
- **Summary**

- **Challenge: use AF in observables (cross sections (σ) (also some amplitudes . . .)) that are** *not infrared safe*

- **Possible** *if:* σ **has a short-distance subprocess. Separate** *IR Safe* **from IR: this is factorization**

- **IR Safe part (short-distance) is calculable in pQCD**

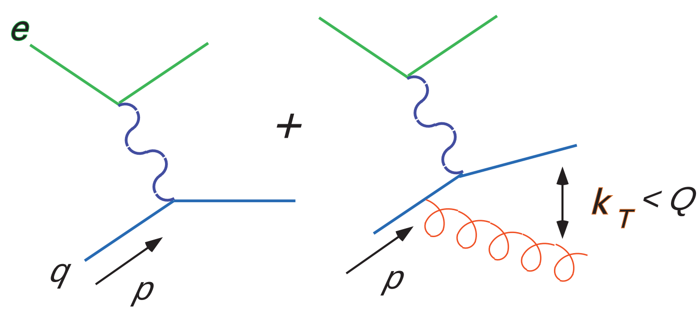
- **Infrared part – example: parton distribution – measureable and universal**

- **Infrared safety – insensitive to soft gluon emission collinear rearrangements**

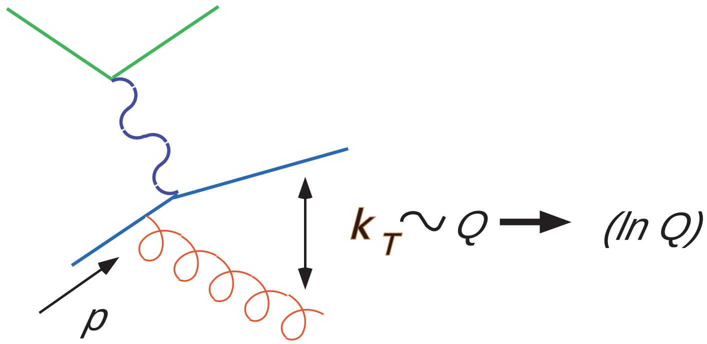
- **Just like Parton Model except in Parton Model the infrared safe part is $\sigma_{\rm Born} \Rightarrow f(x)$ normalized uniquely**

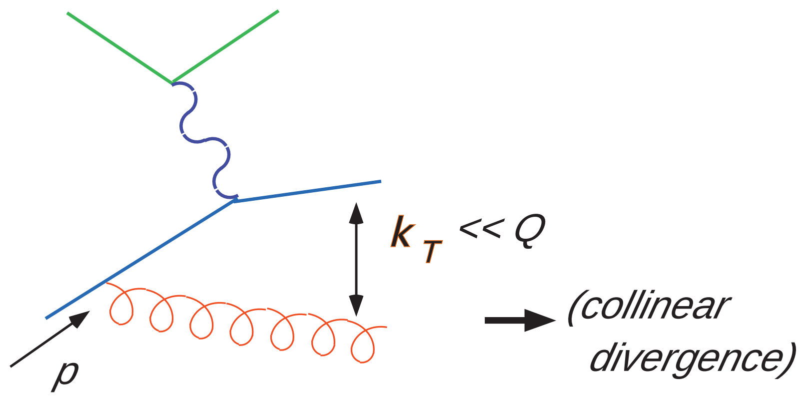
- **In pQCD must define parton distributions more carefully: the factorization scheme**

- **Basic observation: virtual states not truly frozen. Some states fluctuate on scale $1/Q$ . . .**



Short-lived states $\Rightarrow \ln(Q)$





Long-lived states $\Rightarrow$ **Collinear Singularity (IR)**

RESULT: FACTORIZED DIS

$$F_2^{\gamma q}(x, Q^2) = \int_x^1 d\xi\, C_2^{\gamma q}\left(\frac{x}{\xi}, \frac{Q}{\mu}, \frac{\mu_F}{\mu}, \alpha_s(\mu)\right) \times \phi_{q/q}(\xi, \mu_F, \alpha_s(\mu))$$

$$\equiv C_2^{\gamma q}\left(\frac{x}{\xi}, \frac{Q}{\mu}, \frac{\mu_F}{\mu}, \alpha_s(\mu)\right) \otimes \phi_{q/q}(\xi, \mu_F, \alpha_s(\mu))$$

- $\phi_{q/q}$ **has** $\ln(\mu_F/\Lambda_{\mathrm{QCD}})$ **. . .**
- C **has** $\ln(Q/\mu),\ \ln(\mu_F/\mu)$
- **Often pick** $\mu = \mu_F$ **and often pick** $\mu_F = Q$**. So often see:**

$$F_2^{\gamma q}(x, Q^2) = C_2^{\gamma q}\left(\frac{x}{\xi}, \alpha_s(Q)\right) \otimes \phi_{q/q}(\xi, Q^2)$$

- **But we still need to specify what we** *really* **mean by factorization:** *scheme* **as well as** *scale*

- **For this, compute** $F_2^{\gamma q}(x, Q)$

- **Keep** $\mu = \mu_F$ **for simplicity**

- **Factorization in terms of matrix elements:**

$$W_{\mu\nu}^{(\gamma h)} = \frac{1}{8\pi} \sum_{\sigma,X} \langle X|J_\mu(0)|h(p,\sigma)\rangle^* \langle X|J_\nu(0)|h(p,\sigma)\rangle \ (2\pi)^4\,\delta^4(p_X - p - q)$$

$$= \frac{1}{8\pi} \int d^4z\, e^{iq\cdot z} \ \langle h(p,\sigma)\,|\, J^\mu(z) J^\nu(0)\,|\, h(p,\sigma)\rangle$$

$$= \sum_{a=q,\bar{q},G} \int_0^1 d\xi\ C_{\mu\nu}^{\gamma q}\left(\frac{x}{\xi}, \frac{Q}{\mu}, \alpha_s(\mu)\right) \phi_{a/h}(\xi,\mu)$$

$$\phi_{q/h}(\xi,\mu) = \frac{1}{2}\sum_\sigma \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-i\lambda x p\cdot n}\,\langle h(p,\sigma)\mid \bar{q}(\lambda n)\,\frac{n\cdot\gamma}{2}\,q(0)\mid h(p,\sigma)\rangle$$

- n^μ **a light-like vector opposite to** p**-direction.**

- ϕ **is renormalized at** μ **(**$n \cdot A = 0$ **gauge).**

$$\phi_{q/h}(\xi,\mu) = \frac{1}{2}\sum_{\sigma}\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}e^{-i\lambda xp\cdot n}\,\langle h(p,\sigma)\mid \bar{q}(\lambda n)\,\frac{n\cdot\gamma}{2}\,q(0)\mid h(p,\sigma)\rangle$$

$$\sim \langle h(p,\sigma)|\, b^\dagger(\xi p,\lambda)\, b(\xi p,\lambda)\ \mid h(p,\sigma)\rangle + \mathcal{O}(g_s)$$

- **At zeroth order, $\phi_{q/h}(\xi)$ counts the number of quarks at fractional momentum ξ in $h(p,s)$. (spin average in this case)**

- $N(p,\sigma) = b(p,\sigma)^\dagger b(p,\sigma)$**: the "number operator": "takes away" a quark, puts the same one back**

- **This suggests (a very important aside) . . .**

- **The generalized distributions:**

$$\mathcal{F}_{q/h}(x,\zeta,t,\mu) = \frac{1}{2}\sum_s \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-i\lambda x p\cdot n} \langle h(p',\sigma) \mid \bar{q}(\lambda n)\, \frac{n\cdot\gamma}{2}\, q(0) \mid h(p,\sigma)\rangle$$

$$\sim \langle h(p+\zeta p+\vec{\Delta},\sigma)|\, b^\dagger(xp+\Delta,\lambda)\, b(xp,\lambda)\ \mid h(p,\sigma)\rangle + \mathcal{O}(g_s)$$

- **Difference:** $p \to p'$ **for the "out" quark.** $t=\Delta^2$**,** $\zeta = -n\cdot\Delta$.

- **At lowest order: "takes away" a quark at one momentum puts the same kind of quark at a different momentum**

- **Look for a cross section where this is the factorized distribution: amplitude for DVCS:** $q'^{\,2}=0$**,** $q^2=-Q^2$ **large,** $x = Q^2/p\cdot q$ **fixed**

$$T^{\mu\nu}(p',q';p,q) = \frac{1}{2}\sum_\sigma \int d^4z\, e^{-i(q-q')\cdot z}\ \langle h(p',\sigma)\,|\,T\,(J^\mu(z)J^\nu(0)\,|\,h(p,\sigma)\rangle$$

- **"Compute quark-photon scattering"** – *What does this mean?*

- **Must use an** *IR-regulated* **theory**

- **Extract the** *IR Safe part* **then take away the regularization**

- **Let's see how it works . . .**

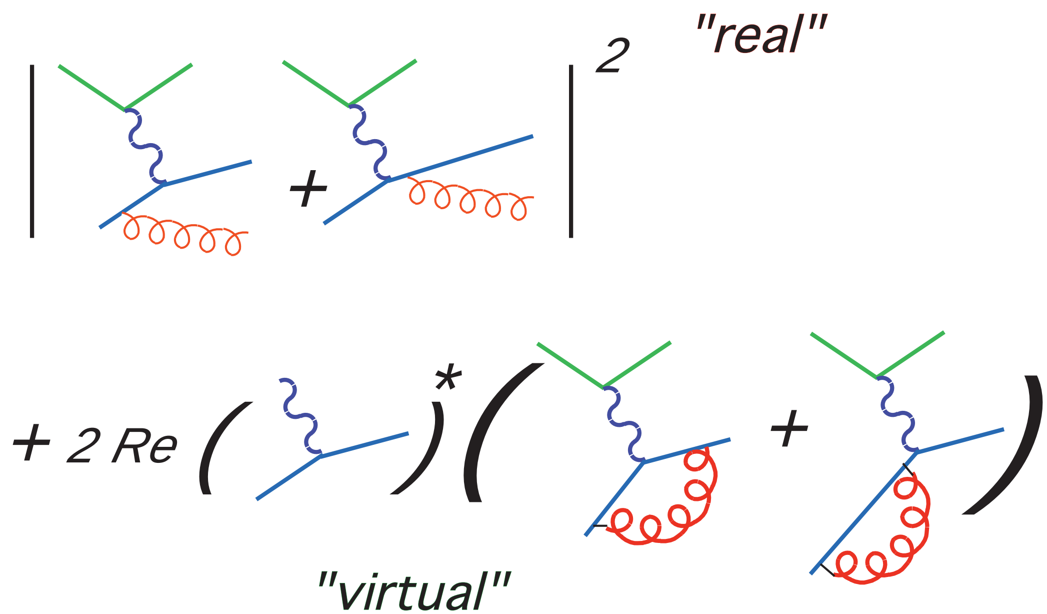
  - **At** *zeroth order – no interactions:*
  - $C^{\gamma q_f (0)} = e_f^2 \, \delta(1 - x/\xi)$
    (Born cross section; parton model)
  - $\phi^{(0)}_{q_f/q_{f'}}(\xi) = \delta_{f f'} \, \delta(1 - \xi)$
    (at zeroth order, momentum fraction conserved)

$$
\begin{aligned}
F_2^{\gamma q_f\,(0)}(x,Q^2) &= \int_x^1 d\xi\; C_2^{\gamma q_f\,(0)}\left(\frac{x}{\xi},\frac{Q}{\mu},\frac{\mu_F}{\mu},\alpha_s(\mu)\right) \\
&\quad \times\; \phi^{(0)}_{q_f/q_f}(\xi,\mu_F,\alpha_s(\mu)) \\
&= e_f^2 \int_x^1 d\xi\; \delta(1-x/\xi)\;\delta(1-\xi) \\
&= e_f^2\; x\,\delta(1-x)
\end{aligned}
$$

– **On to one loop . . .**

$F^{\gamma q}$ AT ONE LOOP: FACTORIZATION SCHEMES

- **Start with F_2 for a $quark$:**



Have to combine final states with different phase space . . .

“Plus Distributions”:

$$\int_0^1 dx \, \frac{f(x)}{(1-x)_+} \quad \equiv \int_0^1 dx \, \frac{f(x)-f(1)}{(1-x)}$$

$$\int_0^1 dx \; f(x)\left(\frac{\ln(1-x)}{1-x}\right)_+ \quad \equiv \int_0^1 dx \; (\, f(x) - f(1)\,) \; \frac{\ln(1-x)}{(1-x)}$$

and so on . . . where

- $f(x)$ **will be parton distributions**
- $f(x)$ **term: real gluon, with momentum fraction** $1-x$
- $f(1)$ **term: virtual, with elastic kinematics**

A Special Distribution
"DGLAP evolution kernel" = "splitting function"

$$P_{qq}^{(1)}(x) = C_F \, \frac{\alpha_s}{2\pi} \, \left[\frac{1+x^2}{1-x}\right]_+$$

- **Will see: P_{qq} a probability per unit log k_T**

Expansion and Result:

$$F_2^{\gamma q}(x,Q^2) \;=\; \int_x^1 d\xi\, C_2^{\gamma q}\left(\frac{x}{\xi},\frac{Q}{\mu},\frac{\mu_F}{\mu},\alpha_s(\mu)\right) \times \phi_{q/q}(\xi,\mu_F,\alpha_s(\mu))$$

$$F_2^{\gamma q_f}(x,Q^2) \;=\; C_2^{(0)}\;\phi^{(0)} + \frac{\alpha_s}{2\pi}\;C^{(1)}\;\phi^{(0)} + \frac{\alpha_s}{2\pi}\;C^{(0)}\;\phi^{(1)} + \dots$$

$$
\begin{aligned}
F_2^{\gamma q_f}(x, Q^2) \;&=\; e_f^2 \left\{ x\ \delta(1-x) \right. \\
&\quad + \frac{\alpha_s}{2\pi}\ C_F \left[\frac{1+x^2}{1-x} \left(\frac{\ln(1-x)}{x} \right) + \frac{1}{4}\,(9-5x) \right]_+ \\
&\quad \left. + \frac{\alpha_s}{2\pi}\ C_F \int_0^{Q^2} \frac{dk_T^2}{k_T^2}\ \left[\frac{1+x^2}{1-x} \right]_+ \ \right\} + \ldots
\end{aligned}
$$

$$
F_1^{\gamma q_f}(x, Q^2) \;=\; \frac{1}{2x}\ \left\{ F_2^{\gamma q_f}(x, Q^2) - C_F \frac{\alpha_s}{\pi}\, 2x \ \right\}
$$

Factorization Scheme

$\overline{\mathrm{MS}}$

$$\phi^{(1)}_{q/q}(x,\mu^2) = \frac{\alpha_s}{2\pi}\ P_{qq}(x)\ \int_0^{\mu^2} \frac{dk_T^2}{k_T^2}$$

With k_T-integral "IR regulated".

Advantage: technical simplicity; not tied to process.

$C^{(1)}(x)_{\overline{\mathrm{MS}}} = (\alpha_s/2\pi)\ P_{qq}(x)\ln(Q^2/\mu^2)$ **+** μ**-independent**

This is the matrix element for the "quark-in-quark distribution":

$$\phi_{q/q}(\xi,\mu) = \frac{1}{2}\sum_s \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-i\lambda x p\cdot n}\, \langle q(p,\sigma)\mid \bar{q}(\lambda n)\,\frac{n\cdot\gamma}{2}\,q(0)\mid q(p,\sigma)\rangle$$

Using the Regulated Theory

and

Getting Parton Distributions for Real Hadrons

- **IR-regulated QCD is not** $REAL$ **QCD**
- BUT **it only differs at low momenta**
- $THUS$ **we can use it for IR Safe functions:** $C_2^{\gamma q}$**, etc.**
- **This enables us to get PDFs for real hadrons . . .**

- **Compute $F_2^{\gamma q}$, $F_2^{\gamma G}$ . . .**

- **Define factorization scheme; find IR Safe C's**

- **Use factorization in the full theory**

$$F_2^{\gamma N} = \sum_{a=q_f,\bar{q}_f,G} C^{\gamma a} \otimes \phi_{a/N}$$

- **Measure F_2; then use the known C's to derive $\phi_{a/N}$**

- **Multiple flavors and cross sections complicate technicalities; not logic (Global Fits)**

$$\mathit{NOW\ HAVE}\ \phi_{a/N}(\xi, \mu^2)$$

USE IT IN ANY OTHER PROCESS THAT FACTORIZES

EVOLUTION

– Q^2-dependence

– **In general,** Q^2/μ^2 **dependence still in** $C_a\left(x/\xi, Q^2/\mu^2, \alpha_s(\mu)\right)$
Choose $\mu = Q$

$$F_2^{\gamma A}(x, Q^2) = \sum_a \int_x^1 d\xi\, C_2^{\gamma a}\left(\frac{x}{\xi}, 1, \alpha_s(Q)\right) \phi_{a/A}(\xi, Q^2)$$

$Q \gg \Lambda_{\rm QCD} \rightarrow$ *compute* C*'s in PT*.

$$C_2^{\gamma a}\left(\frac{x}{\xi}, 1, \alpha_s(Q)\right) = \sum_n \left(\frac{\alpha_s(Q)}{\pi}\right)^n C_2^{\gamma a(n)}\left(\frac{x}{\xi}\right)$$

But still need PDFs at $\mu = Q$**:** $\phi_{a/A}(\xi, Q^2)$ **for different** Q**'s.**

– **Remarkable result: EVOLUTION**

Can use $\phi_{a/A}(x, Q_0^2)$ **to determine**
$\phi_{a/A}(x, Q^2)$ *and hence* $F_{1,2,3}(x, Q^2)$
for any Q *!*

So long at $\alpha_s(Q)$ **is still small**

– **Illustrate by a 'nonsinglet' distribution**

$$F_a^{\gamma \mathrm{NS}} = F_a^{\gamma p} - F_a^{\gamma n}$$

$$F_2^{\gamma \mathrm{NS}}(x, Q^2) = \int_x^1 d\xi \; C_2^{\gamma \mathrm{NS}}\left(\frac{x}{\xi}, \frac{Q}{\mu}, \alpha_s(\mu)\right) \; \phi_{\mathrm{NS}}(\xi, \mu^2)$$

Gluons, antiquarks cancel

At one loop: $C_2^{\mathrm{NS}} = C_2^{\gamma N}$

– **'Mellin' Moments and Anomalous Dimensions**

$$\bar{f}(N) = \int_0^1 dx \ x^{N-1} \ f(x)$$

– **Reduces convolution to a product**

$$f(x) = \int_x^1 dy \ g\left(\frac{x}{y}\right) \ h(y) \rightarrow \bar{f}(N) = \bar{g}(N) \ \bar{h}(N+1)$$

– **Moments applied to NS structure function:**

$$\bar{F}_2^{\gamma \mathrm{NS}}(N, Q^2) = \bar{C}_2^{\gamma \mathrm{NS}}\left(N, \frac{Q}{\mu}, \alpha_s(\mu)\right)\ \bar{\phi}_{\mathrm{NS}}(N, \mu^2)$$

(Note $\bar{\phi}_{\mathrm{NS}}(N, \mu^2) \equiv \int_0^1 d\xi \xi^N f(\xi, \mu^2)$ **here.)**

– $\bar{F}_2^{\gamma \mathrm{NS}}(N, Q^2)$ **is PHYSICAL**

$$\Rightarrow \quad \mu \frac{d}{d\mu}\ \bar{F}_2^{\gamma \mathrm{NS}}(N, Q^2) = 0$$

– **'Separation of variables'**

$$\mu \frac{d}{d\mu} \ln \bar{\phi}_{\rm NS}(N, \mu^2) = -\gamma_{\rm NS}(N, \alpha_s(\mu))$$

$$\gamma_{\rm NS}(N, \alpha_s(\mu)) = \mu \frac{d}{d\mu} \ln \bar{C}_2^{\gamma {\rm NS}}(N, \alpha_s(\mu))$$

– **Because α_s and N are the only variables held in common!**

$$
\begin{aligned}
\mu \frac{d}{d\mu} \ln \bar{\phi}_{\mathrm{NS}}(N,\mu^2) &= -\gamma_{\mathrm{NS}}(N,\alpha_s(\mu)) \\
\gamma_{\mathrm{NS}}(N,\alpha_s(\mu)) &= \mu \frac{d}{d\mu} \ln \bar{C}_2^{\gamma \mathrm{NS}}(N,\alpha_s(\mu))
\end{aligned}
$$

– **Only need to know C's $\Rightarrow \gamma_n$ from IR regulated theory!**

$$\Downarrow$$

Q-DEPENDENCE DETERMINED BY PT

EVOLUTION

THIS WAS HOW WE FOUND OUT QCD IS 'RIGHT'

THIS IS HOW QCD PREDICTS PHYSICS AT NEW SCALES

. . . for hard scattering cross sections, and amplitudes

(Hint: $(1-x^2)/(1-x) = 1 + x \ldots (1-x^k)/(1-x) = \sum_{i=0}^{k-1} x^k$**)**

$$
\begin{aligned}
\gamma_{\mathrm{NS}}^{(1)}(N,\alpha_s) &= \mu \frac{d}{d\mu} \ln \bar{C}_2^{\gamma\mathrm{NS}}\,(N,\alpha_s(Q)) \\
&= \mu \frac{d}{d\mu} \left\{ \,(\alpha_s/2\pi)\; \bar{P}_{qq}(N) \ln(Q^2/\mu^2) + \mu \text{ indep.} \right\} \\
&= -\frac{\alpha_s}{\pi} \int_0^1 dx\; x^{N-1}\; P_{qq}(x) \\
&= -\frac{\alpha_s}{\pi} C_F \int_0^1 dx \left[\left(x^{N-1} - 1 \right) \frac{1+x^2}{1-x} \right] \\
&= -\frac{\alpha_s}{\pi} C_F \left[4 \sum_{m=2}^{N} \frac{1}{m} - 2 \frac{2}{N(N+1)} + 1 \right] \\
&\equiv -\frac{\alpha_s}{\pi}\; \gamma_{\mathrm{NS}}^{(1)}
\end{aligned}
$$

SOLUTION: SCALE BREAKING

$$\mu \frac{d}{d\mu} \bar{\phi}_{\rm NS}(N,\mu^2) = -\gamma_{\rm NS}(N,\alpha_s(\mu))\ \bar{\phi}_{\rm NS}(N,\mu^2)$$

$$\bar{\phi}_{\rm NS}(N,\mu^2) = \bar{\phi}_{\rm NS}(N,\mu_0^2) \times \exp\left[-\frac{1}{2} \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2}\ \gamma_{\rm NS}(N,\alpha_s(\mu)) \right]$$

$$\Downarrow$$

$$\bar{\phi}_{\rm NS}(N,Q^2)\ = \bar{\phi}_{\rm NS}(N,Q_0^2)\ \left(\frac{\ln(Q^2/\Lambda^2_{\rm QCD})}{\ln(Q_0^2/\Lambda^2_{\rm QCD})} \right)^{-2\gamma_N^{(1)}/b_0}$$

Hint:

$$\alpha_s(Q) = \frac{4\pi}{b_0\,\ln(Q^2/\Lambda^2_{\rm QCD})}$$

So also:

$$\bar{\phi}_{\rm NS}(N,Q^2) \quad = \quad \bar{\phi}_{\rm NS}(N,Q_0^2)\,\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-2\gamma_N^{(1)}/b_0}$$

$$\bar{\phi}_{\mathrm{NS}}(N,Q^2) \quad = \quad \bar{\phi}_{\mathrm{NS}}(N,Q_0^2) \ \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-2\gamma_N^{(1)}/b_0}$$

– **‘Mild’ scale breaking**

– **For $\alpha_s \to \alpha_0 \neq 0$, get a power Q-dependence:**

$$\left(Q^2\right)^{\frac{\alpha_0}{2\pi}\,\gamma^{(1)}}$$

– **QCD’s consistency with the Parton Model (73-74)**

$$\mu\frac{d}{d\mu}\;\bar{\phi}_{\rm NS}(N,\mu^2) = -\gamma_N(\alpha_s(\mu))\;\bar{\phi}_{\rm NS}(N,\mu^2)$$

$$\Downarrow$$

$$\mu\frac{d}{d\mu}\;\phi_{\rm NS}(x,\mu^2) = \int_x^1 \frac{d\xi}{\xi}\;P_{\rm NS}\left(\frac{x}{\xi},\alpha_s(\mu)\right)\;\bar{\phi}_{\rm NS}(\xi,\mu^2)$$

Splitting function $\leftrightarrow$ Moments

$$\int_0^1 dx\;x^{N-1}\;P_{qq}(x,\alpha_s) = \gamma_{qq}(N,\alpha_s)$$

BEYOND NONSINGLET COUPLED EVOLUTION

$$\mu \frac{d}{d\mu}\, \phi_{a/A}(x,\mu^2) = \sum_{b=q,\bar{q},G} \int_x^1 \frac{d\xi}{\xi}\, P_{ab}\left(\frac{x}{\xi}, \alpha_s(\mu)\right)\ \bar{\phi}_{b/A}(\xi,\mu^2)$$

Physical Contxt of Evolution

- **Parton Model:** $\phi_{a/A}(x)$ **density of parton** a **with momentum fraction** x**, assumed independent of** Q
- **PQCD:** $\phi_{a/A}(x,\mu)$**: same density, but with transverse momentum** $\leq \mu$

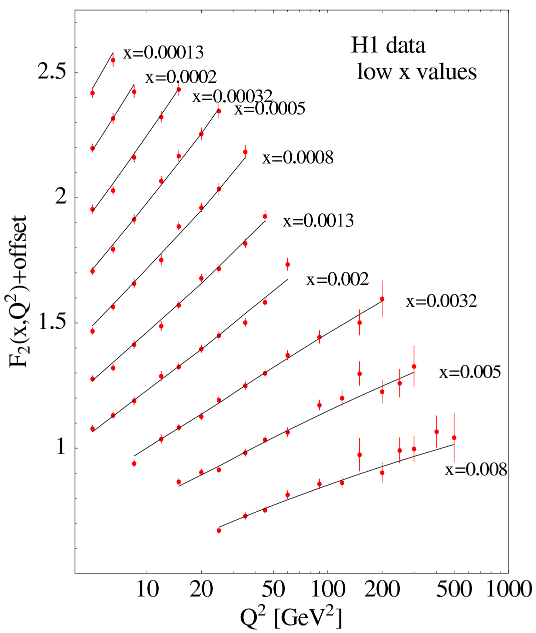
- If there *were* a maximum transverse momentum Q_0, each $\phi_{a/h}(x, Q_0)$ would freeze for $\mu \geq Q_0$

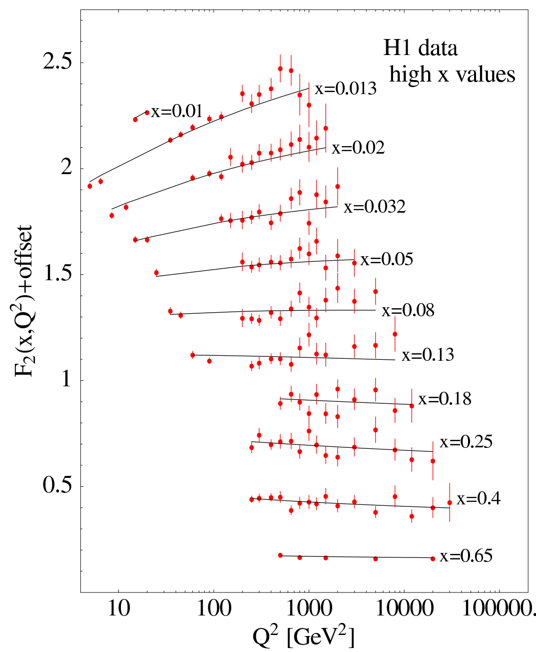
- *Not so* in renormalized PT

- Scale breaking measures the change in the density as maximum transverse momentum increases

- Cross sections we compute still depend on our choice of μ through uncomputed "higher orders" in C and evolution

– Evolution in DIS (with CTEQ6 fits)





Resummation: the Classic Case: Q_T

Every final state from a hard scattering carries the imprint of QCD dynamics from at all distance scales

Resummation extends evolution reasoning to control part of this transition.

- **Look at transverse momentum distribution at order α_s**

$$q(p_1) + \bar{q}(p_2) \to \gamma^*(Q) + g(k)\,,$$

- **Treat this $2 \to 2$ process at lowest order (α_s) "LO" in factorized cross section, so that $\mathbf{k} = -\mathbf{Q}_T$**

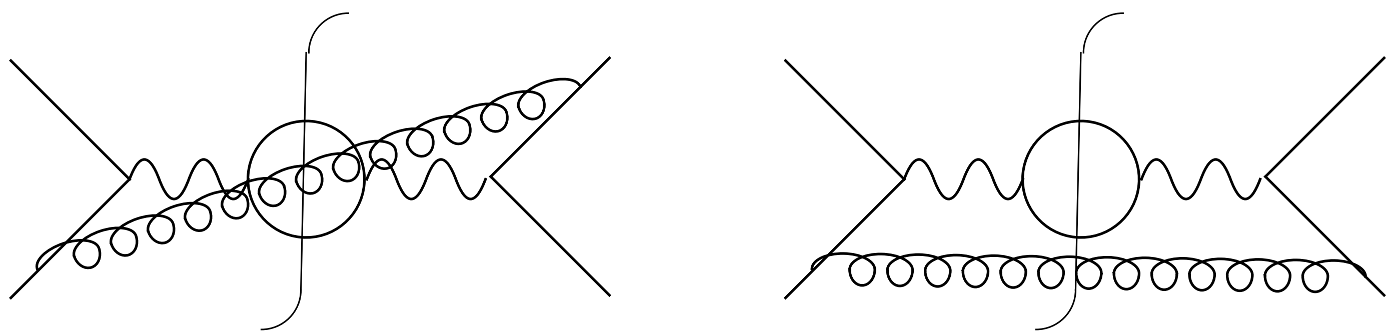
- **Factorized cross section at fixed $\mathbf{Q}_T$:**

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,p_1,p_2)}{dQ^2 d^2\mathbf{Q}_T} = \int_{\xi_1,\xi_2} \sum_{a=q\bar{q}} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu,\xi_1 p_1,\xi_2 p_2,\mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T} \times f_{a/N}(\xi_1,\mu)\, f_{\bar{a}/N}(\xi_2,\mu)$$

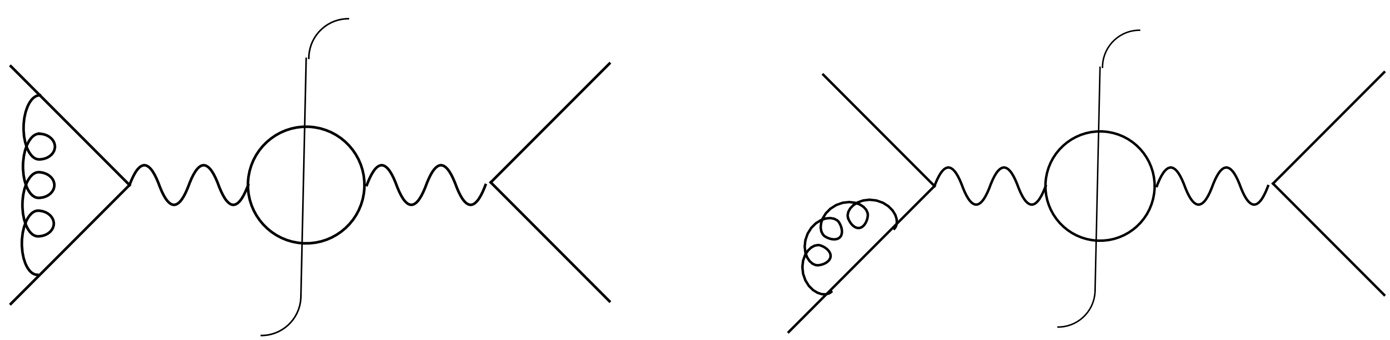
- **μ is the factorization scale that separates IR (f) from UV ($d\hat{\sigma}$) in quantum corrections.**

- **The diagrams at order α_s. Finite for $\mathbf{Q}_T \neq 0$ . . .**

Gluon emission contributes at $Q_T \neq 0$



Virtual corrections contribute only at $Q_T = 0$



$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q}\to\gamma^* g}}{dQ^2\, d^2\mathbf{Q}_T} = \sigma_0 \frac{\alpha_s C_F}{\pi^2} \left(1 - \frac{4\mathbf{Q}_T^2}{(1-z)^2 \xi_1 \xi_2 S}\right)^{-1/2}$$

$$\times \left[\frac{1}{\mathbf{Q}_T^2}\frac{1+z^2}{1-z} - \frac{2z}{(1-z)Q^2}\right]$$

as long as $\mathbf{Q}_T \neq 0$, $z = Q^2/\xi_1\xi_2 S \neq 1$.

Q_T **integral** $\rightarrow \frac{\ln(1-z)}{1-z}$; z **integral** $\rightarrow \frac{\ln \mathbf{Q}_T^2}{\mathbf{Q}_T^2}$.

The leading singularity in $\mathbf{Q}_T$

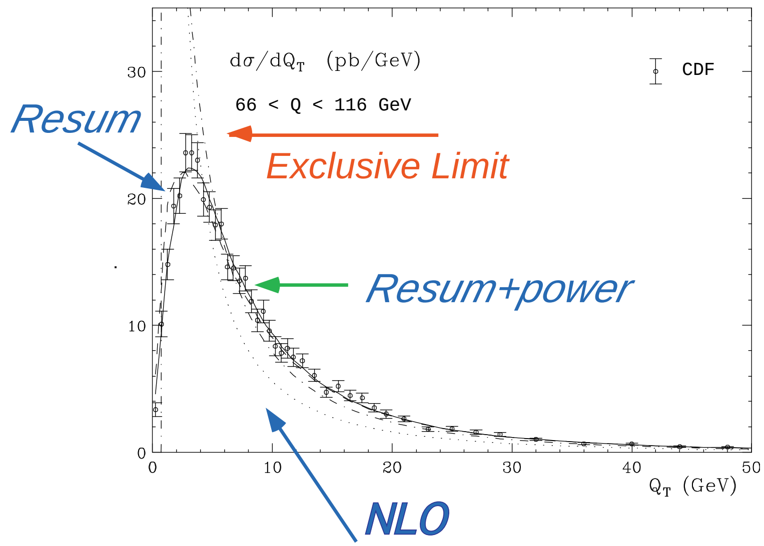
- z **integral:** **If** Q^2/S **not too big, PDFs nearly constant:**

$$\frac{1}{\mathbf{Q}_T^2} \int_{1-Q^2/S}^{1-\mathbf{Q}_T^2/Q^2} \frac{dz}{1-z} = \frac{1}{\mathbf{Q}_T^2} \ln\left[\frac{Q^2}{\mathbf{Q}_T^2}\right]$$

$\Rightarrow$ **Prediction for** Q_T **dependence:**

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,\mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T} = \frac{\alpha_s C_F}{\pi} \frac{1}{\mathbf{Q}_T^2} \ln\left[\frac{Q^2}{\mathbf{Q}_T^2}\right]$$
$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2} f_{a/N}(\xi_1,\mu)\, f_{\bar{a}/N}(\xi_2,\mu)$$

- **Compare to: Z p_T (from Kulesza, G.S., Vogelsang (2002))**



- $\ln Q_T/Q_T$ **works pretty well for large** Q_T

- **But at smaller** Q_T **reach a maximum, then a decrease near "exclusive" limit (parton model kinematics)**

- **Most events are at "low"** $Q_T \ll Q = m_Z$**.**

Getting to $Q_T \ll Q$: Transverse momentum resummation

(Logs of Q_T)/Q_T to all orders

How? Variant factorization and separation of variables

q and $\bar{q}$ "arrive" at point of annihilation with transverse momentum of radiated gluons in initial state.

q and $\bar{q}$ radiate independently (fields don't overlap!).

Final-state QCD radiation too late to affect cross section

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,\mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T}$$

Summarized by: Q_T-factorization:

$$\frac{d\sigma_{NN\to QX}}{dQ\,d^2Q_T} = \int d\xi_1 d\xi_2\; d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} d^2\mathbf{k}_{sT}\, \delta\left(Q_T - k_{1T} - k_{2T} - k_{sT}\right)$$
$$\times\; H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$
$$\times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, k_{1T})\,\mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, k_{2T})\; U_{a\bar{a}}(k_{sT}, n)$$

The $\mathcal{P}'s$: new Transverse momentum-dependent PDFs

Also need U: "soft function" for wide-angle radiation

Symbolically:

$$\frac{d\sigma_{NN\to QX}}{dQ d^2 Q_T} = H \times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, k_{1T})\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, k_{2T})$$
$$\otimes_{\xi_i, k_{iT}}\, U_{a\bar{a}}(k_{sT}, n)$$

We will solve for the k_T dependence of the $\mathcal{P}$'s.

New factorization variables: n^μ apportions gluons k:

$$p_i \cdot k < n\cdot k \;\Rightarrow\; k \in \mathcal{P}_i$$
$$p_a\cdot k, p_{\bar{a}}\cdot k > n\cdot k \;\Rightarrow\; k\in U$$

Convolution in $k_{i,T}$s $\Rightarrow$ Fourier $\mathrm{e}^{i\vec{Q}_T\cdot\vec{b}}$

The factorized cross section in "impact parameter space":

$$\frac{d\sigma_{NN\to QX}(Q,b)}{dQ} = \int d\xi_1 d\xi_2$$
$$\times\ H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$
$$\times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, b)\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, b)\ U_{a\bar{a}}(b, n)$$

Now we can resum by separating variables!

the LHS independent of $\mu_{\rm ren}$, n $\Rightarrow$ two equations

$$\mu_{\rm ren}\frac{d\sigma}{d\mu_{\rm ren}} = 0 \qquad n^\alpha \frac{d\sigma}{dn^\alpha} = 0$$

Method of Collins and Soper, and Sen (1981)

Change in jet must cancel change in (UV) H and (IR) U:

$$p \cdot n \frac{\partial}{\partial p \cdot n} \ln \mathcal{P}(p \cdot n/\mu, b\mu) = G(p \cdot n/\mu) + K(b\mu)$$

G matches H, K matches U. Renormalization indep. of n^μ:

$$\mu \frac{\partial}{\partial \mu} \left[G(p \cdot n/\mu) + K(b\mu) \right] = 0$$

$$\mu \frac{\partial}{\partial \mu} G(p \cdot n/\mu) \;=\; A(\alpha_s(\mu)) \;=\; -\, \mu \frac{\partial}{\partial \mu} K(b\mu)$$

Solve this one first. μ in α_s varies (& α_s need not be small).

$$G(p\cdot n/\mu) + K(b\mu) = G(p\cdot n/\mu) + K(\mu/p\cdot n) - \int_{1/b}^{p\cdot n} \frac{d\mu'}{\mu'} A_a(\alpha_s(\mu'))$$

The consistency equation for the jet becomes

$$p\cdot n \frac{\partial}{\partial p\cdot n} \ln \mathcal{P}(p\cdot n/\mu, b\mu) = G(p\cdot n/\mu) + K(\mu/p\cdot n) - \int_{1/b}^{p\cdot n} \frac{d\mu'}{\mu'} A(\alpha_s(\mu'))$$

Integrate $p\cdot n$ and get double logs in $b \to \alpha_s^n \frac{\ln^{2n-1}(Q/Q_T)}{Q_T}$.

Transformed solution back to Q_T: all the (Logs of Q_T)/Q_T,
Which fits the data; (viz. RESBOS; Yuan, Nadolsky et al.)

$$\frac{d\sigma_{NN\mathrm{res}}}{dQ^2\, d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2b}{(2\pi)^2}\, e^{i\vec{Q}_T\cdot\vec{b}} \exp\left[\, E_{a\bar{a}}^{\mathrm{PT}}(b, Q, \mu)\, \right]$$
$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}\; f_{a/N}(\xi_1, 1/b)\, f_{\bar{a}/N}(\xi_2, 1/b)$$

"Sudakov" exponent links large and low virtuality:

$$E_{a\bar{a}}^{\mathrm{PT}} = -\int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2A_q(\alpha_s(k_T))\, \ln\left(\frac{Q^2}{k_T^2}\right) + 2B_q(\alpha_s(k_T)) \right]$$

With $B = 2(K+G)_{\mu=p\cdot n}$, and lower limit: $1/b$ (NLL)

SUMMARY

- **Specific problems for perturbation theory in QCD**

1. **Confinement**

$$\int \mathrm{e}^{-iq\cdot x} \langle 0|\, T[\phi_a(x)\ldots]\,|0\rangle$$

has no $q^2 = m^2$ pole for ϕ_a that transforms nontrivially under color (confinement)

2. **The pole at $p^2 = m_\pi^2$**

$$\int \mathrm{e}^{-iq\cdot x} \langle 0|\, T[\pi(x)\ldots]\,|0\rangle$$

is not accessible to perturbation theory

- **Response: use infrared safety & asymptotic freedom:**

$$
\begin{aligned}
Q^2\ \hat{\sigma}_{\mathrm{SD}}(Q^2, \mu^2, \alpha_s(\mu)) &= \sum_n c_n(Q^2/\mu^2)\ {\alpha_s}^n(\mu) + \mathcal{O}\left(1/Q^p\right) \\
&= \sum_n c_n(1)\ {\alpha_s}^n(Q) + \mathcal{O}\left(1/Q^p\right)
\end{aligned}
$$

- **What can we really calculate? PT for color singlet operators**

  – $\int \mathrm{e}^{-iq\cdot x}\langle 0|\, T[J(x)J(0)\ldots]\,|0\rangle$ **for color singlet currents**

    $\mathrm{e}^+\mathrm{e}^-$ **total . . . no QCD in initial state**

– **Jet cross sections are from matrix elements also:**

$$\lim_{R\to\infty}\int dx_0 \int d\hat{n}\, S(\hat{n})\, \mathrm{e}^{-iq\cdot y}\langle 0|\, J(0)T[\hat{n}_i T_{0i}(x_0, R\hat{n})J(y)]\,|0\rangle$$

Where the operator T_{0i} measures momentum flow

"Weight" $S(\hat{n})$ introduces no new dimensional scale

Short-distance dominated if all $d^kS/d\hat{n}^k$ bounded

Individual final states have IR divergences, but these cancel in sum over collinear splitting/merging and soft parton emission because they respect energy flow

But what of the initial state? (viz. parton model)

- **Factorization**

$$Q^2\sigma_{\rm phys}(Q,m) = \omega_{\rm SD}(Q/\mu,\alpha_s(\mu)) \otimes \phi_{\rm LD}(\mu,m) + \mathcal{O}\left(1/Q^p\right)$$

  - μ = **factorization scale;** m= **IR scale**
  - **New physics in** $\omega_{\rm SD}$; $\phi_{\rm LD}$ = f **and/or** D **"universal"**

  - **ep DIS inclusive,** ${\rm pp} \to$ **jets,** $Q\bar{Q}$, $\pi(p_T)$, **DVCS . . .**

  - **Exclusive limits:** ${\rm e}^+{\rm e}^- \to {\rm JJ}$ **as** $m_J \to 0$

- **Whenever there is factorization, there is evolution**

$$0 = \mu \frac{d}{d\mu} \ln \sigma_{\rm phys}(Q, m)$$

$$\mu \frac{d \ln(\phi \,{\rm or}\; D)}{d\mu} = -P(\alpha_s(\mu)) = -\mu \frac{d \ln \omega}{d\mu}$$

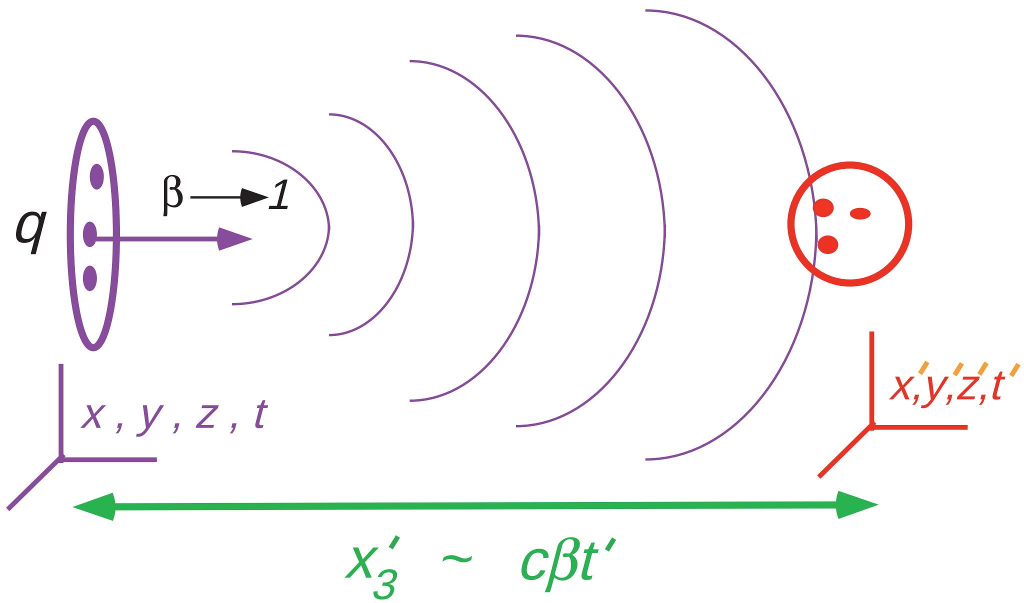
PDF ϕ <u>or</u> Fragmentation D

- **Wherever there is evolution there is resummation**

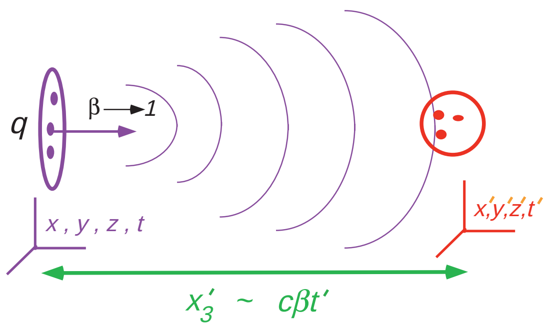
$$\ln \sigma_{\rm phys}(Q, m) = \exp \left\{ \int_q^Q \frac{d\mu'}{\mu'} P\left(\alpha_s(\mu')\right) \right\}$$

- **Appendix: Basis of Factorization proofs:**
  - **(1) $\omega_{\rm SD}$ incoherent with long-distance dynamics**
  - **(2) Mutual incoherence when $v_{\rm rel} = c$:**
    Jet-jet factorization.
  - **(3) Wide-angle soft radiation sees only total color flow:**
    jet-soft factorization.
  - **(4) Dimensionless coupling and renormalizability**
    $\Leftrightarrow$ no worse that logarithmic divergence in the IR:
    suppression even by a fractional power $\Rightarrow$ finiteness

– **Hadron-Hadron Factorization** **Heuristic, classical argument:**



$$\Delta \equiv \beta c t' - x'_3$$



field	x frame	x' frame
scalar	$\frac{q}{\lvert\vec{x}\rvert}$	$\frac{q}{(x_T^2+\gamma^2\Delta^2)^{1/2}}$
gauge	$A^0(x) = \frac{q}{\lvert\vec{x}\rvert}$	$A'^0(x') = \frac{q\gamma}{(x_T^2+\gamma^2\Delta^2)^{1/2}}$
field strength	$E_3(x) = \frac{-q}{\lvert\vec{x}\rvert^2}$	$E_3'(x') = \frac{-q\gamma\Delta}{(x_T^2+\gamma^2\Delta^2)^{3/2}}$

– Classical: Lorentz contracted fields of incident particles don't overlap until the moment of the scattering, creation of heavy particle, etc.!

– Initial-state interactions decouple from the hard process

– Summarized by multiplicative factors:
parton distributions

– Evolution of partons to jets/hadrons too late to know details of the hard scattering

– Summarized by multiplicative factors:
fragmentation functions

– "Left-over" cross section for hard scattering is IR safe